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Cyclic Tournaments and Cooperative Majority Voting: A Solution*

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Abstract. A new solution concept is axiomatically characterized for tournaments that represent cooperative majority voting. The predicted set of outcomes lies inside the top-cycle set, the uncovered set, and the Banks set.

The solution to a cooperative majority-voting game is the set of feasible alternatives that a majority might make a contract to support. Identifying that set is a problem when the relation of majority preference is cyclic. The problem is often simplified by assuming, as I do here, that the feasible alternatives are finite in number and the relation of majority preference is a tournament (which means there are not ties). Given these assumptions, a growing literature has examined three potential solutions to majority-voting games, cooperative and noncooperative: the *top-cycle set*, a comparatively large set that can have Pareto-inefficient elements; the *uncovered set*, a Pareto-efficient subset of the top-cycle set; and the *Banks set*, a subset of the uncovered set. Each of these sets is perforce nonempty, and each is definable in terms of the majority-preference tournament. More familiar cooperative-solution concepts (the core, the V-solution, the bargaining set) lack the former trait and sometimes the latter.

The difficulty we face in evaluating all such concepts is that parliamentary procedure does not structure cooperative voting as it does noncooperative voting. The final outcome of noncooperative committee voting, sincere or strategic, can be defined in terms of an extensive form (or simpler arboreal structure) determined by the agenda (Farquharson 1969; McKelvey and Niemi 1978; Moulin 1979; Ordeshook and Schwartz 1987). But under cooperative voting, the agenda is a mere formality: a majority make a contract to support a particular alternative before voting begins, possibly before the agenda is set. One might evaluate a solution

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concept empirically. But in the absence of a theoretical rationale, a host of supportive test results would leave us with a generalization more than an explanation. An axiomatic characterization – a set of reasonable-looking axioms uniquely satisfied by a proposed solution – would provide a rationale. But to the extent that the axioms owed their appeal (as collective-choice axioms often do) to simplicity, intuitive naturalness, or normative plausibility, the characterization theorem would leave open the question why behavior should be expected to conform to the axioms.

My approach is axiomatic, but my axioms are designed to serve as a substitute for the invisible extensive form: they relate predicted outcomes to available moves in the underlying game. These axioms are uniquely satisfied by the *tournament equilibrium set,* a nonempty subset of the top-cycle set, the uncovered set, and the Banks set.

Although I motivate the axioms in terms of cooperative majority voting, the formal problem of choice from cyclic tournaments has other interpretations, to which the tournament equilibrium set may be applicable. Instead of interpreting the tournament as majority preference, we can interpret it more generally as the relation of x to y when x is preferred to y by some group of voters who together have the power to overturn y in favor of x. Such power need not be possessed by all or only majorities. Nor need players be voters. They can be two candidates for some office, each trying to choose a platform preferred to his opponent's by a majority of the electorate. The solution is then the set of platforms that the winner might adopt. The formal problem also has interpretations unrelated to collective choice, including choice by individuals with cyclic preferences (May 1954; Schwartz 1970, 1972, 1986, pp 104–112; Packard 1975a, b, 1982) and the scoring of real games (Moon 1968; Litvakov and Vol'skiy 1985).

Section 1 contains some formal preliminaries, Sect. 2 the definitions of the top-cycle set, the uncovered set, and the Banks set, Sect. 3 the axioms, and Sect. 4 the definition of the tournament equilibrium set. The characterization theorem is proved in Sect. 5, and the tournament equilibrium set is related to the top-cycle, uncovered, and Banks sets in Sect. 6. Section 7 concludes with some open questions.

1. Formal Preliminaries

Assume that P is a tournament on a nonempty set A: P is a binary relation on A that is asymmetric and connected in A (for all distinct $x, y \in A$, either xPy or yPx). Elements of A are denoted x, y, z, etc.; finite, nonempty subsets of A are denoted α, β, γ.

Interpret P as the majority-preference relation of a committee, governed by majority rule, that chooses cooperatively from finite subsets of A. Thanks to McGarvey (1953), any tournament on A is the majority-preference relation of some conceivable committee.

A *choice function* is any function C of finite, nonempty subsets of A such that $C(\alpha) \subseteq \alpha$ for all α. Choice functions are denoted C, C_1, C_2, etc. Solution concepts will be formulated as definitions of choice functions.

2. The Top-cycle Set, the Uncovered Set, and the Banks Set

The choice functions G, U, and B, whose respective values are the top-cycle set, the uncovered set, and the Banks set, are defined as follows:

$$G(\boldsymbol{\alpha}) = \{x \in \boldsymbol{\alpha} | x \text{ is a Condorcet winner in } \boldsymbol{\alpha} \text{ or a member of a top } P\text{-cycle in } \boldsymbol{\alpha}\},$$

where

x is a *Condorcet winner* in $\boldsymbol{\alpha}$ iff $x \in \boldsymbol{\alpha}$ and xPy for all $y \neq x$ in $\boldsymbol{\alpha}$,

and

$\boldsymbol{\beta}$ is a *top P-cycle* in $\boldsymbol{\alpha}$ iff $\boldsymbol{\beta} \subseteq \boldsymbol{\alpha}$, $\boldsymbol{\beta}$ is a P-cycle ($\boldsymbol{\beta} = \{x_1, \ldots, x_n\}$, and $x_1Px_2P \ldots Px_nPx_1$ for some n and $x_1, \ldots, x_n$), and xPy for all $x \in \boldsymbol{\alpha}$ and $y \in \boldsymbol{\alpha} - \boldsymbol{\beta}$.

$$U(\boldsymbol{\alpha}) = \{x \in \boldsymbol{\alpha} | \text{ nothing covers } x \text{ in } \boldsymbol{\alpha}\} ,$$

where

y covers x in $\boldsymbol{\alpha}$ iff $y, x \in \boldsymbol{\alpha}$, yPx, and for all $z \in \boldsymbol{\alpha}$, if xPz then yPz.

$$B(\boldsymbol{\alpha}) = \{x | x \text{ is the first coordinate of some maximum } P\text{-chain in } \boldsymbol{\alpha}\},$$

where

a *P-chain in* $\boldsymbol{\alpha}$ is a vector $(x_1, \ldots, x_n)$ $(n \geq 1)$ such that $x_1, \ldots, x_n \in \boldsymbol{\alpha}$ and x_iPx_j whenever $i<j$, $i, j = 1, 2, \ldots, n$; it is a *maximum* one iff no longer P-chain in $\boldsymbol{\alpha}$ contains $x_1, \ldots, x_n$.

G is the function defined by the GETCHA and GOCHA conditions of Schwartz (esp. 1986, pp 139–155); although not generally equivalent, these two conditions are equivalent when P is a tournament. Deb (1977) pointed out that the GOCHA set and, more specifically, the top-cycle set for tournaments can have Pareto-inefficient elements. Fishburn (1977) and Miller (1980, 1983) independently introduced U and proved that $U(\boldsymbol{\alpha})$ is always a Pareto-efficient subset of $G(\boldsymbol{\alpha})$. Banks (1985) defined B, drawing on an idea of Shepsle and Weingast (1984), and showed that $B(\boldsymbol{\alpha})$ is always a subset, sometimes proper, of $U(\boldsymbol{\alpha})$.

All three choice functions have figured prominently in studies of noncooperative majority voting (Campbell 1977; Miller 1980; Shepsle and Weingast 1984; Banks 1985; Schwartz 1986, pp 191–223; McKelvey 1986; Ordeshook and Schwartz 1987). They have also been examined in the abstract, in part with the idea that they might represent reasonable solution concepts for cooperative majority voting (Schwartz 1972, 1977; Ferejohn and Grether 1974; Bordes 1976; Miller 1980; Moulin 1986; Miller et al. 1986; Banks and Bordes 1988).

3. Axioms

The axioms are conditions on a choice function C, interpreted so that $C(\boldsymbol{\alpha})$ comprises those alternatives which a majority might make a *final contract* to choose from $\boldsymbol{\alpha}$, not merely a tentative contract that gives way to another and perhaps another in a preliminary process of recontracting. Were such a process to begin, it would end with the choice of a Condorcet winner if one existed. Otherwise it would continue indefinitely and end somewhat arbitrarily, but the final choice – the choice stipulated by the somewhat arbitrary final contract – would have to belong to $C(\alpha)$, and there is no smaller set to which it would have to belong. To make this interpretation of C clearer, we must look closely at the recontracting process.

Suppose $x \in \alpha$ and yPx for some $y \in \alpha$. Then any tentative contract to choose x from α can be upset by a contract to choose something else. What else? Not necessarily y. For y might be one of several alternatives in α, say $y_1, \ldots, y_k$, that bear P to x. Could just any one of these y_i replace x in the imagined recontracting process? To suppose so is to suppose, in effect, that anything in α might be selected as a possible replacement for x, then pitted against x in a contest in which each committee member votes for the alternative he prefers. But such a procedure would be noncooperative. Because members act cooperatively, they will cooperatively choose a replacement for x: some majority will make a contract to replace x with one of the y_i, a contract that is final in the process of replacing x, though perhaps only tentative in the larger process of choosing from α. Given our interpretation of C, this means that an alternative can replace x in the recontracting process iff it belongs to $C(\{y_1, \ldots, y_k\})$. When choosing a replacement for x, committee members are constrained to choose from $\{y_1, \ldots, y_k\}$ rather than α as a whole because no majority would agree to replace x unless they preferred the replacement to x. Still, committee members choose from $\{y_1, \ldots, y_k\}$ as they would from any set: their choice belongs to $C(\{y_1, \ldots, y_k\})$.

To formalize this idea, define:

$$xD(C, \alpha)y \text{ iff } y \in \alpha \text{ and } x \in C(\{t \in \alpha | tPy\}) \ .$$

My suggestion, then, is that a tentative contract to choose x from α can be upset by a contract to choose y just when $yD(C, \alpha)x$.

Now define:

β is *C-retentive* in α iff $\phi \neq \beta \subseteq \alpha$ and nothing in $\alpha - \beta$ bears $D(C, \alpha)$ to anything in β.

The recontracting process, once having entered or begun within a C-retentive subset, can never depart, but it *can* depart from any *non-C*-retentive subset. So unless $C(\alpha)$ were C-retentive, we could not predict (as we wish to do) that the committee's choice from α will belong to $C(\alpha)$. Hence our first axiom:

RETENTIVENESS. $C(\alpha)$ is C-retentive in α.

What we expect of $C(\alpha)$ is not only that the recontracting process would never depart from $C(\alpha)$ but that $C(\alpha)$ have no proper subset in which the process must end up, else our prediction that the committee's choice will belong to $C(\alpha)$ would not be the strongest we could make. So C should also satisfy:

PREDICTIVE STRENGTH. There is no proper subset β of $C(\alpha)$ such that β is C-retentive in α but $C(\alpha) - \beta$ is not.

If the recontracting process began (by chance or design) in a C-retentive subset, it would never depart, so the committee's choice would belong to that subset. Therefore, C must satisfy our final axiom:

INCLUSIVENESS. If β is C-retentive in α then some member of β belongs to $C(\alpha)$.

Although Retentiveness, Predictive Strength, and Inclusiveness constrain C, they are couched in terms of replaceability in the recontracting process, and that relation was defined in terms of C itself. Despite this apparent weakness, the three axioms are together satisfied by exactly one choice function.

4. The Tournament Equilibrium Set

I call that function TEQ. $TEQ(\alpha)$ is the tournament equilibrium set corresponding to α. Here is the definition:

> $TEQ(\alpha)=$ the union of minimum TEQ-retentive subsets of α ,

where, for any C:

> β is a *minimum C-retentive subset* of α iff β is C-retentive in α but no proper subset of β is C-retentive in α.

Although it defines TEQ in terms of TEQ-retentiveness, this definition is not circular. It is recursive, the recursion variable being the cardinality of α: assuming that $TEQ(\beta)$ has been defined for all β smaller than α, the definition uniquely determines $TEQ(\alpha)$. *Proof:* If α is a singleton then α is C-retentive in itself for every C, and thus $TEQ(\alpha)=\alpha$. Now suppose that α has more than one member and $TEQ(\beta)$ has been defined for all β smaller than α. Then if $y\in\alpha$ and $\{t\in\alpha|tPy\}\neq\phi$, $TEQ(\{t\in\alpha|tPy\})$ has been defined. Hence, so has $D(TEQ,\alpha)=\{(x,y)|y\in\alpha$ and $x\in TEQ(\{t\in\alpha|tPy\})\}$ and therewith TEQ-retentiveness in α and $TEQ(\alpha)$. □

TEQ shares two virtues with G, U, and B: it always chooses something, and it chooses the Condorcet winner and it alone when one exists.

Theorem 1. $TEQ(\alpha)\neq\phi$.

Proof. For any C, α is C-retentive in itself. Since α is finite, it follows that α must have at least one *minimum* C-retentive subset, which, by the definition of C-retentiveness, must be nonempty. Hence, the union of such subsets is nonempty, so $TEQ(\alpha)\neq\phi$. □

Theorem 2. *If x is a Condorcet winner in α then* $TEQ(\alpha)=\{x\}$.

Proof. By induction on the cardinality of α. For every $y\neq x$ in α, $x\in\{t\in\alpha|tPy\}$ and xPw for all $w\in\{t\in\alpha|tPy\}$, whence $TEQ(\{t\in\alpha|tPy\})=\{x\}$ by inductive hypothesis, and thus $xD(TEQ,\alpha)y$. But by hypothesis of the theorem and the asymmetry of P, $\{t\in\alpha|tPx\}=\phi$, so nothing bears $D(TEQ,\alpha)$ to x. Because $xD(TEQ,\alpha)y$ for all $y\neq x$ in α whereas $zD(TEQ,\alpha)x$ for no z, $\{x\}$ is the only minimum TEQ-retentive subset of α. Therefore, $TEQ(\alpha)=\{x\}$. □

Theorems 1 and 2 make it easier to reckon $TEQ(\alpha)$. So does this theorem:

Theorem 3. *If $xPyPzPx$ then* $TEQ(\{x,y,z\})=\{x,y,z\}$.

Proof. Let $\alpha=\{x,y,z\}$. By Theorem 1 and the asymmetry of P,

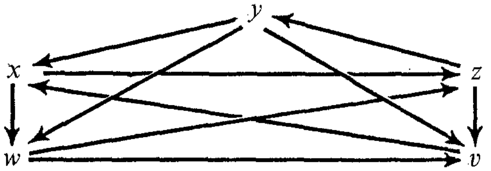
$$TEQ(\{t\in\alpha|tPx\})=TEQ(\{z\})=\{z\}\ ,$$

$$TEQ(\{t\in\alpha|tPy\})=TEQ(\{x\})=\{x\}\ ,\quad\text{and}$$

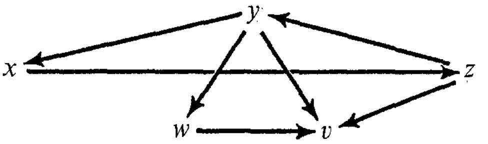
$$TEQ(\{t\in\alpha|tPz\})=TEQ(\{y\})=\{y\}\ .$$

So $zD(TEQ,\alpha)x$, $xD(TEQ,\alpha)y$, and $yD(TEQ,\alpha)z$. Hence, α is the only minimum TEQ-retentive subset of α. □

To illustrate, suppose $\alpha=\{x,y,z,w,v\}$ and P has the form:



Then $TEQ(\{t\in\alpha|tPx\})=TEQ(\{y,v\})=\{y\}$ by Theorem 2, so y and it alone bears $D(TEQ,\alpha)$ to x. Similarly, x and it alone bears $D(TEQ,\alpha)$ to z, and y and it alone bears $D(TEQ,\alpha)$ to w. But $TEQ(\{t\in\alpha|tPy\})=TEQ(\{z\})=\{z\}$ by Theorem 1, so z and it alone bears $D(TEQ,\alpha)$ to y. Finally, $TEQ(\{t\in\alpha|tPv\})=TEQ(\{y,z,w\})=\{y,z,w\}$ by Theorem 3, so y, z, w, and they alone bear $D(TEQ,\alpha)$ to v. In short $D(TEQ,\alpha)$ has the following form:



Because $\{x,y,z\}$ is the only minimum TEQ-retentive subset of α, $TEQ(\alpha)=\{x,y,z\}$.

5. Characterization Theorem

Here is the promised rationale behind TEQ:

Theorem 4. *TEQ is the one and only choice function that satisfies Retentiveness, Predictive Strength, and Inclusiveness.*

Proof. Any α must have at least one minimum TEQ-retentive subset by Theorem 1. But any union of TEQ-retentive subsets of α must itself be TEQ-retentive in α. So $TEQ(\alpha)$ is always TEQ-retentive in α. That is, TEQ satisfies Retentiveness.

To prove that TEQ satisfies Predictive Strength, suppose for *reductio* that some $\beta\subset TEQ(\alpha)$ is TEQ-retentive in α but $TEQ(\alpha)-\beta$ is not. Then $xD(TEQ,\alpha)y$ for some $x\in\alpha-(TEQ(\alpha)-\beta)$ and $y\in TEQ(\alpha)-\beta$. Because $TEQ(\alpha)$ is TEQ-retentive in α, $x\notin\alpha-TEQ(\alpha)$, so $x\in TEQ(\alpha)$, and thus, since $x\in\alpha-(TEQ(\alpha)-\beta)$, $x\in\beta$. But $y\in TEQ(\alpha)-\beta$, so $y\in\gamma-\beta$ for some minimum TEQ-retentive subset γ of α. Because

$xD(TEQ,\alpha)y$ and γ is TEQ-retentive, $x \in \gamma$. Hence, $x \in \gamma \cap \beta$, and thus, since $y \in \gamma - \beta$, $\gamma \cap \beta \subset \gamma$. Because β and γ are TEQ-retentive in α, however, so is $\gamma \cap \beta$, contrary to the minimum TEQ-retentiveness of γ.

To prove that TEQ satisfies Inclusiveness, suppose β is TEQ-retentive in α. Because β is finite, some subset γ of β must be a minimum TEQ-retentive subset of α, hence a subset of $TEQ(\alpha)$. By definition of retentiveness, $\gamma \neq \phi$. Hence, some member of $\gamma \subseteq \beta$ belongs to $TEQ(\alpha)$.

To prove uniqueness, let C_1 and C_2 satisfy the three axioms; I will show, by induction on the cardinality of α, that $C_1(\alpha) = C_2(\alpha)$. By the definition of D,

$$D(C_1, \alpha) = \{(x, y) | y \in \alpha \quad \text{and} \quad x \in C_1(\{t \in \alpha | tPy\})\} \quad \text{and}$$

$$D(C_2, \alpha) = \{(x, y) | y \in \alpha \quad \text{and} \quad x \in C_2(\{t \in \alpha | tPy\})\} \ .$$

But for all $y \in \alpha$, if $\{t \in \alpha | tPy\} \neq \phi$ then $C_1(\{t \in \alpha | tPy\}) = C_2(\{t \in \alpha | tPy\})$ by inductive hypothesis. Hence, $D(C_1, \alpha) = D(C_2, \alpha)$, and thus, for all β,

β is C_1-rententive in α iff β is C_2-retentive in α.

But C_1 and C_2 both satisfy Retentiveness. Therefore, $C_1(\alpha)$ and $C_2(\alpha)$ are both C_2-retentive in α, whence so is $C_1(\alpha) \cap C_2(\alpha)$ if $C_1(\alpha) \cap C_2(\alpha) \neq \phi$. By Inclusiveness, however, because $C_1(\alpha)$ is C_2-retentive in α, some member of $C_1(\alpha)$ belongs to $C_2(\alpha)$; that is, $C_1(\alpha) \cap C_2(\alpha) \neq \phi$. Thus, $C_1(\alpha) \cap C_2(\alpha)$ is indeed C_2-retentive in α. Hence, by Predictive Strength, either $C_2(\alpha) - (C_1(\alpha) \cap C_2(\alpha))$ also is C_2-retentive in α, or else $C_1(\alpha) \cap C_2(\alpha)$ is not a proper subset of $C_2(\alpha)$. But if $C_2(\alpha) - (C_1(\alpha) \cap C_2(\alpha))$ were C_2-retentive in α than it would be C_1-retentive as well, and thus, by Inclusiveness, some member of $C_2(\alpha) - (C_1(\alpha) \cap C_2(\alpha))$ would belong to $C_1(\alpha)$, which is absurd. It follows that $C_1(\alpha) \cap C_2(\alpha)$ is not a proper subset of $C_2(\alpha)$. So $C_1(\alpha) \cap C_2(\alpha) = C_2(\alpha)$. By a similar argument, $C_1(\alpha) \cap C_2(\alpha) = C_1(\alpha)$. Hence, $C_1(\alpha) = C_2(\alpha)$. □

6. *TEQ*, *G*, *U*, and *B*

The tournament equilibrium set is a subset of the other proposed solutions defined in Sect. 2.

Theorem 5. *$TEQ(\alpha)$ is a subset of $G(\alpha)$, $U(\alpha)$, and $B(\alpha)$.*

Proof. As we saw in Sect. 2, $B(\alpha) \subseteq U(\alpha) \subseteq G(\alpha)$. So it suffices to show, by induction on the cardinality of α, that $TEQ(\alpha) \subseteq B(\alpha)$. Suppose $x \in TEQ(\alpha)$. Then x belongs to some minimum TEQ-retentive subset β of α.

Banks (1985) shows that $x \in B(\alpha)$ if x is a Condorcet winner in α. Otherwise, $\{t \in \alpha | tPx\} \neq \phi$ by the connexity of P, whence $TEQ(\{t \in \alpha | tPx\}) \neq \phi$ by Theorem 1, so $zD(TEQ, \alpha)x$ for some $z \in \alpha$. Therefore, $\beta \neq \{x\}$, whence $\beta - \{x\} \neq \phi$, and thus, if $xD(TEQ, \alpha)w$ for no $w \in \beta - \{x\}$ then $\beta - \{x\}$ would be TEQ-retentive in α, contrary to the minimum TEQ-retentiveness of β. Consequently, $xD(TEQ, \alpha)w$ for some $w \in \beta - \{x\}$, so $x \in TEQ(\{t \in \alpha | tPw\})$.

By inductive hypothesis, then, $x \in B(\{t \in \alpha | tPw\})$, so that $x = x_1$ is the first coordinate of some maximum P-chain $(x_1, \ldots, x_k)$ $(k \geq 1)$ in $\{t \in \alpha | tPw\}$. Because

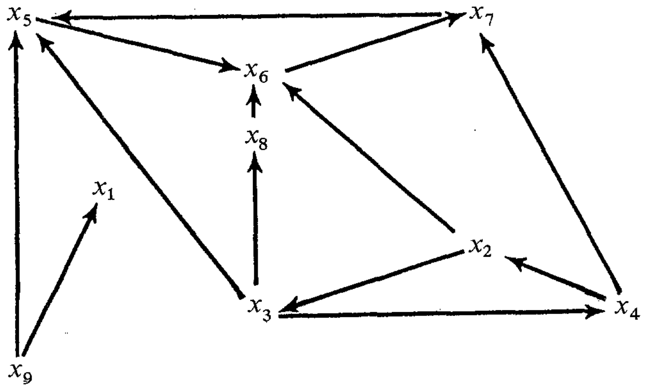
$x_i P w$ for all i, $(x_1, \ldots, x_k, w)$ is a P-chain in α as well. And because α is finite, there must exist a maximum P-chain $(y_1, \ldots, y_m)$ in α containing $x_1, \ldots, x_k, w$. Suppose that y_1 did not belong to $\{x_1, \ldots, x_k, w\}$. Then since $(y_1, \ldots, y_m)$ is a P-chain containing $x_1, \ldots, x_k, w$, y_1 would bear P to $x_1, \ldots, x_k, w$, so $(y_1, x_1, \ldots, x_k)$ would be a P-chain in $\{t \in \alpha | tPw\}$. But that is impossible because $(x_1, \ldots, x_k)$ is a *maximum* P-chain in $\{t \in \alpha | tPw\}$. Hence, $y_1 \in \{x_1, \ldots, x_k, w\}$. Since, however, y_1 is the first coordinate of a P-chain containing $x_1, \ldots, x_k, w$, y_1 must bear P to every other member of $\{x_1, \ldots, x_k, w\}$. But $x_1 = x$ is the only member of this set that bears P to every other member. It follows that $y_1 = x$, so x is the first coordinate of a maximum P-chain in α. That is, $x \in B(\alpha)$. □

This theorem leaves open the possibility that $TEQ(\alpha) = B(\alpha)$. But in the example of Sect. 4, $TEQ(\alpha) = \{x, y, z\}$ whereas $B(\alpha) = U(\alpha) = \{x, y, z, w\}$ and $G(\alpha) = \alpha$. So $TEQ(\alpha)$ sometimes is a proper subset of $G(\alpha)$, $U(\alpha)$, and $B(\alpha)$.

In this case, U and B yield the same set as TEQ if we allow reiteration:

$$U(U(\alpha)) = B(B(\alpha)) = \{x, y, z\} = TEQ(\alpha) = TEQ(TEQ(\alpha)) \ .$$

But that does not always happen. Suppose $\alpha = \{x_1, \ldots, x_9\}$ and P has the following form, where, in the absence of any arrow, *height on the page* represents P:



Then

$$G(\alpha) = \alpha \ ,$$

$$U(\alpha) = \{x_1, \ldots, x_8\} \ ,$$

$$U(U(\alpha)) = \{x_1, \ldots, x_7\} = U(U(U(\alpha))) \ ,$$

$$B(\alpha) = \{x_2, \ldots, x_8\} \ , \quad \text{and}$$

$$B(B(\alpha)) = \{x_2, \ldots, x_7\} = B(B(B(\alpha))) \ , \quad \text{but}$$

$$TEQ(\alpha) = \{x_5, x_6, x_7\} = TEQ(TEQ(\alpha)).$$

7. Discussion

Open questions abound. In the examples above and in all others that I have examined, TEQ has three properties:

If there is no Condorcet winner in α then $TEQ(\alpha)$ is a $D(TEQ, \alpha)$ – cycle (not, for example, the union of two nonoverlapping $D(TEQ, \alpha)$ – cycles).

For every $x \in \alpha - TEQ(\alpha)$, $yD(TEQ, \alpha)x$ for some $y \in TEQ(\alpha)$ (so that no more than one step of recontracting ever is needed to jump from $\alpha - TEQ(\alpha)$ into $TEQ(\alpha)$).

$TEQ(\alpha) = TEQ(TEQ(\alpha))$ (a closure property obviously shared by G but not, we have lately seen, by U or B).

The problem in each case is to find a proof or counter-example.

Such problems would be easier to solve we had a more direct (nonrecursive) definition of TEQ in term of P. Following Dutta (1988), call β a *covering set* for α iff $U(\beta) = \beta \subseteq \alpha$ and $x \in U(\beta \cup \{x\})$ for no $x \in \alpha - \beta$, and define the choice function DUT so that $DUT(\alpha)$ is the smallest covering set for α (of which a unique one must exist, as Dutta proves). In all cases that I have examined, $DUT(\alpha) = TEQ(\alpha)$. The problem again is to find a proof or counter-example.

An open question more conceptual than mathematical concerns the generalization from tournaments to arbitrary directed graphs – binary relations on A that are asymmetric but not necessarily connected in A. Let P be any such relation. With ties allowed, there now are several natural ways to define replaceability in the recontracting process:

$$xD_1(C, \alpha)y \quad \text{iff} \quad y \in \alpha \quad \text{and} \quad x \in C(\{t \in \alpha | tPy\}) \ .$$

$$xD_2(C, \alpha)y \quad \text{iff} \quad y \in \alpha \quad \text{and} \quad x \in C(\{t \in \alpha | t \neq y \text{ and not } yPt\}) \ .$$

$$xD_3(C, \alpha)y \quad \text{iff} \quad xD_2(C, \alpha)y \quad \text{but not} \quad yD_2(C, \alpha)x \ .$$

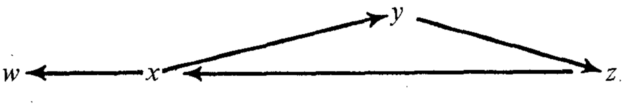
$$xD_4(C, \alpha)y \quad \text{iff} \quad xD_2(C, \alpha)y \quad \text{and} \quad xPy \ .$$

$$xD_5(C, \alpha)y \quad \text{iff} \quad xD_2(C, \alpha)y \quad \text{and} \quad zPy \text{ for some } z \in \alpha \ .$$

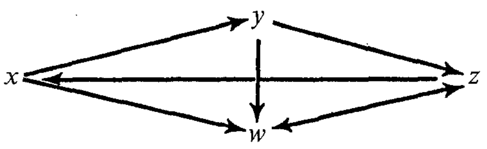
$$xD_6(C, \alpha)y \quad \text{iff} \quad xD_5(C, \alpha)y \quad \text{but not} \quad yD_5(C, \alpha)x \ .$$

Define $TEQ_1, \ldots, TEQ_6$ in terms of $D_1, \ldots, D_6$ as TEQ was defined in terms of D. Observe that the definition of D_1 is the same as that of D. Observe, too, that $D_1, \ldots, D_6$ are all the same if P is connected in A.

Because $D_2(C, \alpha)$ and $D_5(C, \alpha)$ are not necessarily asymmetric, I have included their asymmetric factors, $D_3(C, \alpha)$ and $D_6(C, \alpha)$, in the list. Suppose $\alpha = \{x, y, z, w\}$ and P has the following form, where the absence of any arrow now represents a tie:

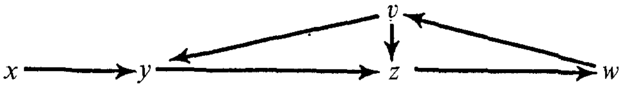


Then $D_2(TEQ_2, \alpha)$ and $D_5(TEQ_5, \alpha)$ share this form:



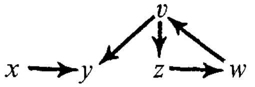
Owing to the two-headed arrow, neither relation is asymmetric. Moreover, $TEQ_2(\alpha)=TEQ_5(\alpha)=\alpha$, an intuitively unacceptable solution. By deleting the two-headed arrow, we depict the corresponding asymmetric relations, $D_3(TEQ_3,\alpha)$ and $D_6(TEQ_6,\alpha)$, and find that $TEQ_3(\alpha)=TEQ_6(\alpha)=\{x,y,z\}$, which seems reasonable.

But in other cases, TEQ_3 yields counter-intuitive results, as do TEQ_1 and TEQ_4. Suppose $\alpha=\{x,y,z,w,v\}$ and P has the form:

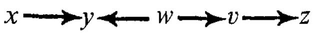


Then $D_1(TEQ_1,\alpha)$, $D_3(TEQ_3,\alpha)$, $D_4(TEQ_4,\alpha)$, and $D_6(TEQ_6,\alpha)$ have the following forms:

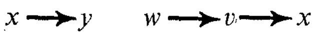
$D_1(TEQ_1,\alpha)$



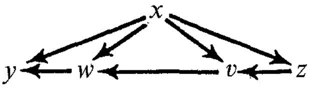
$D_3(TEQ_3,\alpha)$



$D_4(TEQ_4,\alpha)$



$D_6(TEQ_6,\alpha)$



Consequently,

$$TEQ_1(\alpha)=\{x,v,z,w\}\ ,$$

$$TEQ_3(\alpha)=TEQ_4(\alpha)=\{x,w\}\ ,\quad \text{and}$$

$$TEQ_6(\alpha)=\{x\}\ .$$

Only $TEQl_6(\alpha)$ is a reasonable solution.

These examples make a strong case against $TEQ_1,\ldots,TEQ_5$ but only a weak case for TEQ_6. A proof that $TEQ_6(\alpha)$ is always a subset of the *GOCHA* set from α (Schwartz 1972, 1986: Chap. 6) would strengthen that case. So would a proof that $TEQ_6(\alpha)$ is always a subset of $U(\alpha)$ and therewith the Pareto set from α. Still needed would be a theoretical rationale for interpreting recontracting in terms of D_6.

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